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## ► To cite this version:

Bruno Brouard, Denis Lafarge, Jean-François Allard. A general method of modelling sound propagation in layered media. *Journal of Sound and Vibration*, 1995, 183 (1), pp.129-142. 10.1006/jsvi.1995.0243 . hal-02518410

**HAL Id: hal-02518410**

**<https://univ-lemans.hal.science/hal-02518410>**

Submitted on 12 Mar 2024

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# **A general method of modelling sound propagation in layered media**

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A general method of modelling acoustic fields in stratified media which include elastic solid, fluid and porous layers is presented. The simplicity and the versatility of the method is illustrated with several examples.

## **1. INTRODUCTION**

Sound propagation of plane waves in layers of fluid and elastic solid can be modelled by using transfer matrices [1, 2]. A similar representation has been developed for the case of porous materials having an elastic frame [3]. Configurations including plates, impervious screens and layers of air and of porous media have also been studied [4, 5]. In these cases, the surface impedance was predicted for configurations with an impervious backing at the rear face, and the transmission coefficient for configurations with a semi-infinite layer of air at the rear face. These combinations of materials are generally non-locally reacting, and their behaviour more or less depends on the dimensions and the boundary conditions at the edges. Nevertheless, interesting trends can be obtained from modelling of infinite samples subjected to incident plane waves. Using the transfer matrices related to each layer considerably simplifies the modelling, but accounting for the properties of the acoustic field at the boundaries of the different layers remains difficult. It is shown in the following sections that a simple standard form of representation can be carried out for a large variety of stratified materials, if interface matrices carrying sufficient information concerning the acoustic field at the boundaries between the layers are used.

## **2. PROPAGATION THROUGH A STRATIFIED MATERIAL CONSIDERED AS A LINEAR BLACK BOX**

A stratified material is represented in Figures 1(a) and (b). A plane wave impinges upon the material at an angle of incidence  $\theta$ , on the left-hand side of Figure 1(a), and on the right-hand side in Figure 1(b). At the right-hand side in Figure 1(a), the material is in contact with a semi-infinite layer of air, and the acoustic field reduces to a progressive wave. In Figure 1(b), the same condition exists at the left-hand side. In order to simplify the notations, the pressure amplitude of the incident wave is unity in Figures 1(a) and 1(b) and the amplitudes of the transmitted waves are  $T_b$  and  $T_a$ . These coefficients are the transmission coefficients in opposite directions. The pressure amplitudes  $R_a$  and  $R_b$  are the reflection coefficients at each side. Invariance with respect to translations along  $x_1$  is required in order that there exist well-defined reflection and transmission coefficients. Translational invariance in the normal direction  $x_3$  will also be assumed for each layer,

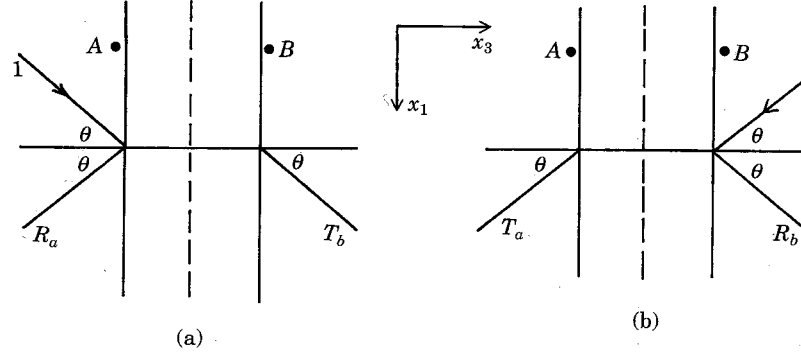


Figure 1. Stratified materials in air. A plane wave impinges upon the material at (a) the left-hand side and (b) the right-hand side.

so that a simple and well-known transfer matrix describes the propagation in a given layer (the purpose of the present paper is not to derive the more complex transfer matrices that would describe depth-dependent layers).

Let  $p(A)$ ,  $p(B)$ ,  $v_3^f(A)$ ,  $v_3^f(B)$  be the pressures and the  $x_3$  components of velocity at  $A$  and  $B$ . The same linear equation relates  $(p(A), v_3^f(A))$  to  $(p(B), v_3^f(B))$  in Figures 1(a) and (b). In matrix form, this equation can be written as

$$\begin{pmatrix} p(A) \\ v_3^f(A) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p(B) \\ v_3^f(B) \end{pmatrix}, \quad (1)$$

which becomes

$$\begin{pmatrix} 1 + R_a & \frac{\cos \theta}{Z_c} \\ (1 - R_a) & \frac{\cos \theta}{Z_c} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} T_b & \frac{\cos \theta}{Z_c} \\ T_b & \frac{\cos \theta}{Z_c} \end{pmatrix},$$

$$\begin{pmatrix} T_a & \frac{\cos \theta}{Z_c} \\ -T_a & \frac{\cos \theta}{Z_c} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 + R_b & \frac{\cos \theta}{Z_c} \\ (R_b - 1) & \frac{\cos \theta}{Z_c} \end{pmatrix}, \quad (2, 3)$$

where  $Z_c$  is the characteristic impedance of air. From equations (2) and (3), the following expressions can be obtained for  $T_a$  and  $T_b$ :

$$T_b = \frac{2Z_c^{-1} \cos \theta}{c + dZ_d^{-1} \cos \theta + Z_c^{-1} \cos \theta(a + bZ_c^{-1} \cos \theta)}, \quad T_a = (ad - bc)T_b. \quad (4, 5)$$

It has been shown [1] that

$$ad - bc = 1 \quad (6)$$

for stratified materials including fluid and viscoelastic layers, and porous layers in the context of Biot theory. The transmission coefficients  $T_a$  and  $T_b$  are equal, and equation (6) indicates that the linear black box is reciprocal.

Given the properties of the internal layers, the coefficients  $a$ ,  $b$ ,  $c$  and  $d$  are uniquely determined by the angular frequency  $\omega$  and projection  $k_1$  of the wave number parallel to the surface:

$$a = a(\omega, k_1), \quad b = b(\omega, k_1), \quad \text{etc}, \dots, \quad (7)$$

and  $k_1$  is given by

$$k_1 = (\omega/c) \sin \theta, \quad (8)$$

where  $c$  is the adiabatic speed of sound in air. It may be noticed that uniform mean flow, parallel to the  $x_1$  direction, can be present in either one or both of the semi-infinite layers of air, with only minor changes in the above algebra, provided that the properties of the linear black box are unaffected. This is briefly discussed in the Appendix. Changes are as follows.

Clearly, the same projection  $k_1$  arises in the different layers, but, in presence of a mean flow  $V$ , equation (8) becomes

$$k_1 = \frac{\omega}{c} \frac{\sin \theta}{1 + (V/c) \sin \theta} . \quad (9)$$

Thus, the angle  $\theta$  is different in the two semi-infinite layers if the mean flow  $V$  is not the same. The characteristic impedance, however, is independent of  $V$ . Consequently, equations (2) and (3) still apply with, possibly, two distinct values of  $\theta$ , say  $\theta_a$  and  $\theta_b$ , and the coefficients  $a$ ,  $b$ ,  $c$  and  $d$  are obtained by setting in equation (7) the value of  $k_1$  given in equation (9) (determined with either the index  $a$  or  $b$  on  $V$  and  $\theta$ ). Evidently, equation (6) still holds since the values of  $\omega$  and  $k_1$  are arbitrary. The new versions of equations (4) and (5) are given in the Appendix.

A general method of calculating  $R_a$ ,  $R_b$  and  $T_a = T_b = T$ , is given in the following sections for the case where no mean flow is present. As outlined above and in the Appendix, generalization to the case of uniform external flow would not be difficult, if one ignores penetration effects and the boundary layers at the front and rear surfaces. Uniform mean flow inside an internal layer of air could also be considered (with the same simplifying hypothesis). However, it is likely to be the case that between two plane boundaries the steady flow to be introduced is of the Poiseuille type. The properties of the layer would not be invariant in the normal direction, and as noted previously, such depth-dependent layers will not be considered.

### 3. THE MATRIX REPRESENTATION OF PLANE ACOUSTIC FIELDS IN LAYERED MEDIA

### 3.1. TRANSFER MATRICES RELATED TO FLUID, SOLID ELASTIC AND POROUS LAYERS

Several layers of a stratified material are represented in Figure 2. The stratified material is in contact with air at both faces, and the acoustic field is created by an incident plane

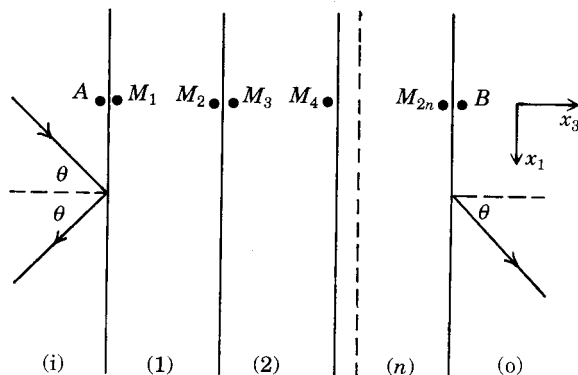


Figure 2. Several layers in contact in a stratified material.

wave at the left-hand side of the stratified material. The plane of incidence is the plane  $x_1, x_3$ , and  $\theta$  is the angle of incidence. The  $x_1$  component  $k_1$  of the wavenumber vector in each layer is given by

$$k_1 = k \sin \theta, \quad (10)$$

where  $k$  is the wavenumber in air. Let  $v_3^s(M), v_1^s(M)$  be the  $x_3$  and  $x_1$  velocity components of the solid at  $M$  (in the case of an elastic solid, or of the frame of a porous material). Similarly, let  $v_3^f(M), v_1^f(M)$  be the  $x_3$  and  $x_1$  velocity components of the fluid at  $M$  (in the case of a fluid, or of the fluid in a porous material). In the examples to be studied, the fluid will be air. Let  $\sigma_{33}^s, \sigma_{13}^s$  and  $\sigma_{33}^f, \sigma_{13}^f$  be the normal and the tangential stresses. The stresses are related to forces acting per unit area of material. In a fluid, they become

$$\sigma_{13}^f = 0, \quad \sigma_{33}^f = -p, \quad (11, 12)$$

where  $p$  is the pressure. In a porous medium, equation (12) becomes

$$\sigma_{33}^f = -\Phi p, \quad (13)$$

where  $\Phi$  is the porosity. If the layer is a fluid layer, the acoustic field is determined everywhere when a pair  $(p(M), v_3^f(M))$  is known. A  $2 \times 2$  matrix relates the pressure and the  $x_3$  velocity component at the right- and left-hand sides of the layer ( $M_2$  and  $M_1$  for the first layer of the stratified material of Figure 2): i.e.,

$$V^f(M_1) = [T^f]V^f(M_2), \quad (14)$$

where

$$V^f(M_2) = [p(M_2), v_3^f(M_2)]^T. \quad (15)$$

In equation (15), the superscript T indicates the transposition line→column. The elements of  $[T^f]$  are

$$\begin{aligned} T_{11}^f &= \cos k_3 L, & T_{12}^f &= (\omega \rho / k_3) j \sin k_3 L, \\ T_{21}^f &= (k_3 / \omega \rho) j \sin k_3 L, & T_{22}^f &= \cos k_3 L, \end{aligned} \quad (16)$$

where  $\rho$  is the density,  $L$  the thickness of the layer, and the symbol  $j$  represents  $\sqrt{-1}$ . The  $x_3$  component  $k_3$  of the wavenumber vector in the fluid is given by

$$k_3 = (k_f^2 - k_1^2)^{1/2}, \quad (17)$$

where  $k_f$  is the wavenumber in the fluid.

Similar matrices exist for elastic solid layers [2]. The difference with the previous case is that two kinds of waves, rotational and dilatational, can propagate in the medium, and four quantities are needed to characterize the acoustic field. If the first layer in Figure 2 is an elastic solid layer, two vectors  $V^s(M_1)$  and  $V^s(M_2)$  are related by a  $4 \times 4$  transfer matrix  $[T^s]$ : i.e.,

$$V^s(M_1) = [T^s]V^s(M_2), \quad (18)$$

where

$$V^s(M) = [v_1^s, v_3^s, \sigma_{33}^s, \sigma_{13}^s]^T. \quad (19)$$

The elements of  $[T^s]$  have been calculated by Folds and Loggins [2] for the time dependence  $\exp(-j\omega t)$ , a permutation of  $V^s(M_1)$  and  $V^s(M_2)$  in equation (18) and a different numbering of the components of  $V^s$ .

In the context of Biot theory [6], three different kinds of waves can propagate in a porous material. The transfer matrix  $[T^p]$  is a  $6 \times 6$  matrix, given in reference [3], and the vectors  $V^p(M)$  related to this matrix are given by

$$V^p(M) = [v_1^s, v_3^s, v_3^f, \sigma_{33}^s, \sigma_{13}^s, \sigma_{33}^f]^T. \quad (20)$$

### 3.2. INTERFACE MATRICES

For the two layers 1 and 2 in contact, as represented in Figure 2, the points  $M_2$  and  $M_3$  are close to each other at each side of the boundary. The interface matrices relate the acoustic fields at  $M_2$  and  $M_3$ , the relation depending on the nature of the two layers.

For two fluids, the boundary conditions are

$$V^f(M_2) = V^f(M_3). \quad (21)$$

Two interface matrices  $[I_{ff}]$  and  $[J_{ff}]$  exist, such that

$$[I_{ff}]V^f(M_2) + [J_{ff}]V^f(M_3) = 0. \quad (22)$$

$[I_{ff}]$  and  $[J_{ff}]$  must be opposite, and  $[I_{ff}]$  can be any diagonal matrix, the  $2 \times 2$  unit matrix for instance.

For two solids, equation (21) is rewritten as

$$V^s(M_2) = V^s(M_3). \quad (23)$$

The matrices  $[I_{ss}]$  and  $[J_{ss}]$  are opposite, and  $[I_{ss}]$  can be the  $4 \times 4$  matrix.

For two porous layers, the boundary conditions, as given in reference [7], are

$$\sigma_{13}^s(M_2) = \sigma_{13}^s(M_3), \quad \sigma_{33}^f(M_2) + \sigma_{33}^s(M_2) = \sigma_{33}^f(M_3) + \sigma_{33}^s(M_3), \quad (24, 25)$$

$$v_1^s(M_2) = v_1^s(M_3), \quad v_3^s(M_2) = v_3^s(M_3), \quad (26, 27)$$

$$\Phi_1(v_3^f(M_2) - v_3^s(M_2)) = \Phi_2(v_3^f(M_3) - v_3^s(M_3)), \quad \sigma_{33}^f(M_2)/\Phi_1 = \sigma_{33}^f(M_3)/\Phi_2, \quad (28, 29)$$

where  $\Phi_1$  and  $\Phi_2$  are the porosities at  $M_2$  and  $M_3$ . Equation (22) can be rewritten as

$$[I_{pp}]V^p(M_2) + [J_{pp}]V^p(M_3) = 0, \quad (30)$$

where  $[I_{pp}]$  is the  $6 \times 6$  unit matrix, and  $[J_{pp}]$  is given by

$$[J_{pp}] = - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 - \phi_2/\Phi_1 & \Phi_2/\Phi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & (1 - \Phi_1/\Phi_2) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_1/\Phi_2 \end{bmatrix}. \quad (31)$$

For a fluid and a solid, the boundary conditions are

$$-p(M_2) = \sigma_{33}^s(M_3), \quad 0 = \sigma_{13}^s(M_3), \quad v_3^f(M_2) = v_3^s(M_3). \quad (32-34)$$

Equations (32)–(34) can be rewritten as

$$[I_{fs}]V^f(M_2) + [J_{fs}]V^s(M_3) = 0, \quad (35)$$

where

$$[I_{fs}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [J_{fs}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (36, 37)$$

For a fluid and a porous layer, the boundary conditions are

$$v_3^f(M_2) = (1 - \Phi_2)v_3^s(M_3) + \Phi_2 v_3^f(M_3), \quad -\Phi_2 p(M_2) = \sigma_{33}^f(M_3), \quad (38, 39)$$

$$-(1 - \Phi_2)p(M_2) = \sigma_{33}^s(M_3), \quad 0 = \sigma_{13}^s(M_3). \quad (40, 41)$$

These equations can be rewritten as

$$[I_{f,p}]V^f(M_2) + [J_{f,p}]V^p(M_3) = 0, \quad (42)$$

where

$$[I_{f,p}] = \begin{bmatrix} 0 & -1 \\ \phi_2 & 0 \\ (1 - \Phi_2) & 0 \\ 0 & 0 \end{bmatrix}, \quad [J_{f,p}] = \begin{bmatrix} 0 & (1 - \Phi_2) & \Phi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad (43, 44)$$

For an elastic solid and a porous layer, the boundary conditions are

$$v_3^s(M_2) = v_2^s(M_3) = v_3^f(M_3), \quad v_1^s(M_2) = v_1^s(M_3), \quad (45, 46)$$

$$\sigma_{13}^s(M_2) = \sigma_{13}^s(M_3), \quad \sigma_{33}^s(M_2) = \sigma_{33}^f(M_3) + \sigma_{33}^s(M_3). \quad (47, 48)$$

These equations can be rewritten as

$$[I_{s,p}]V^s(M_2) + [J_{s,p}]V^p(M_3) = 0, \quad (49)$$

where

$$[I_{s,p}] = - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [J_{s,p}] = + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \quad (50, 51)$$

The interface matrices  $[I_{p,s}]$ ,  $[J_{p,s}]$ ,  $[I_{p,f}]$ ,  $[J_{p,f}]$  and  $[I_{s,f}]$ ,  $[J_{s,f}]$ , can easily be obtained from the previous equations.

### 3.3. PLANE ACOUSTIC FIELD IN A LAYERED MATERIAL

Let  $V(M_1)$ ,  $V(M_2)$ ,  $\dots$ ,  $V(M_{2n})$  be the vectors which define the acoustic field at the boundaries of the layers in Figure 2. It may be noticed that a simple product of transfer and interface matrices cannot generally be used to calculate the transfer matrix of equation (1), because non-square interface matrices appear on the left-hand sides of equations (35), (42) and (49). These vectors are related by the following equations:

$$[I^{i,1}]V(A) + [J^{i,1}][T^{(1)}]V(M_2) = 0, \quad [I^{1,2}]V(M_2) + [J^{1,2}][T^{(2)}]V(M_4) = 0, \\ [I^{n,o}]V(M_{2n}) + [J^{n,o}]V(B) = 0. \quad (52-54)$$

This set of equations can be rewritten as

$$[D]V_D = 0, \quad (55)$$

where

$$V_D = [p(A), v^f(A), V(M_2), \dots, V(M_{2n}), p(B), v^f(B)]^T. \quad (56)$$

Acoustic propagation in the stratified material of Figure 2 is completely defined by equations (52)–(54), but more information is needed to define the acoustic field. For example, if the layer of air at the right-hand side of the material is semi-infinite, the impedance at  $B$  is given by

$$Z_b = p(B)/v_3^f(B) = Z_c/\cos \theta. \quad (57)$$

and equation (55) becomes

$$\begin{bmatrix} & & [D] \\ 0 & \cdots & 0 & -1 & Z_b \end{bmatrix} V_D = [D'] V_D = 0. \quad (58)$$

Other simple conditions, for example a layer of air of finite thickness instead of a semi-infinite layer, can easily be introduced in the same way. Let  $N$  be the dimension of  $V_D$ . The matrix  $[D]$  has  $N - 2$  lines. It is related to the fact that the acoustic field is completely determined if, and only if, a supplementary condition is added to equation (58); for example, the pressure amplitude at  $A$ . The impedance  $Z_a$  of the layered material of Figure 1(a) (incident field at the left-hand side of the material) can be calculated by adding to the set defined by equation (58) the equation

$$p(A) - Z_a v_3^f(A) = 0. \quad (59)$$

The determinant of the matrix  $[D'']$  given by

$$[D''] = \begin{bmatrix} -1 & Z_a & 0 & \cdots & 0 \\ & & [D] & & \\ 0 & \cdots & 0 & -1 & Z_b \end{bmatrix}, \quad (60)$$

is equal to zero, the  $Z_a$  is given by

$$Z_a = -|D'_1|/|D'_2|, \quad (61)$$

where  $|D'_1|$  (resp.  $|D'_2|$ ) is the determinant of the matrix  $[D']$ , the first column (resp. the second column) having been removed. The calculation of  $Z_b$  for the case of an incident plane wave impinging upon the right-hand side of the stratified material (Figure 1(b)) is very similar.

The calculation of the transmission coefficient  $T$  can be carried out in the following way. The set of equations defined by the matrix equation (58) can be completed, instead of by equation (59), by the equation

$$p(A)[T/(1 + R_a)] - p(B) = 0. \quad (62)$$

The determinant of this new set of equations is equal to zero and  $T$  is given by

$$T = +(1 + R_a)|D'_{N-1}|/|D'_1|, \quad (63)$$

where  $|D'_{N-1}|$  is the determinant of  $[D']$ , the  $(N - 1)$ th column having been removed.

It may be noticed that the proposed method is very general. More complicated stratified materials can be considered, including anisotropic layers or composite layers. The relevant assumption is that of linear propagation through each layer, and the boundaries between the layers.

The modelling described in this section is relatively straightforward, and has been implemented in a general program which can also be used to predict the surface impedance of stratified materials bonded on to a rigid backing. Two very different illustrations, which show the versatility of the program, are given in the following sections.



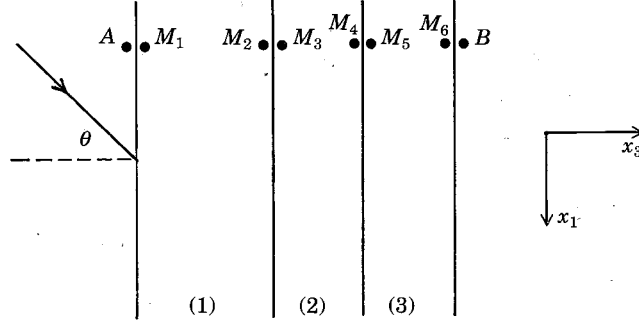


Figure 3. A viscoelastic material (2) covered by a thin aluminium layer (3) is bonded onto a plate of aluminium (1).

#### 4. FIRST EXAMPLE

##### 4.1. TRANSMISSION IN A STRATIFIED MATERIAL MADE UP OF A VISCOELASTIC MATERIAL SANDWICHED BY TWO LAYERS OF ALUMINIUM

The material is represented in Figure 3. A viscoelastic material covered by a thin aluminium layer is bonded on to a plate of aluminium, in order to increase the damping of the plate. The behaviour of the material, for the case of vibrations induced by a plane wave incident on the left-hand side, can easily be described by the general model developed previously. The matrix  $[D]$  and the vector  $V$  of equation (55) are

$$[D] = \begin{bmatrix} [I_{f,s}^i] & [J_{f,s}^i][T^{(1)}] & 0 & 0 & 0 \\ 0 & [I] & [T^{(2)}] & 0 & 0 \\ 0 & 0 & [I] & [T^{(3)}] & 0 \\ 0 & 0 & 0 & [I_{s,f}^o] & [J_{s,f}^o] \end{bmatrix}, \quad (64)$$

$$V = [p(A), v_s^i(A), V(M_2), V(M_4), V(M_6), p(B), v_s^i(B)], \quad (65)$$

where  $[I]$  is an unit matrix.

The parameters which characterize the three layers are given in Table 1. A loss angle for aluminium larger than the effective has been used to account for the finite dimension effect and the fastening.

The transmission coefficient can be calculated by equation (63). The transmission loss is given by

$$F = -10 \log_{10} \int_0^{\theta_{max}} |T(\theta)|^2 \cos \theta \sin \theta d\theta, \quad (66)$$

where  $\theta_{max}$  generally is taken equal to  $80^\circ$  [8]. Other values for  $\theta_{max}$  can be chosen [9]. We have used  $\theta_{max} = 90^\circ$ , which gives the best agreement between prediction and measurement.

TABLE 1  
*Parameters for the three layers of the first example*

Parameter	Layer 1	Layer 2	Layer 3
Thickness (m)	$1.6 \times 10^{-3}$	$0.5 \times 10^{-4}$	$1.5 \times 10^{-4}$
Density (kg/m <sup>3</sup> )	$2.800 \times 10^3$	$1.01 \times 10^3$	$2.8 \times 10^3$
Shear modulus (Pa)	$2.4 \times 10^{10} + j4 \times 10^8$	$2 \times 10^6(1 + j)$	$2.4 \times 10^{10} + j4 \times 10^8$
Poisson ratio	0.34	0.499	0.34
Loss factor	$1.6 \times 10^{-2}$	1	$1.6 \times 10^{-2}$

The measured and predicted transmission losses are represented in Figure 4 for the stratified material of the first example. The agreement between prediction and measurement is good in the high frequency range, but the predicted transmission loss is too small at frequencies lower than 8 kHz.

#### 4.2. INTERNAL DAMPING IN THE STRATIFIED MATERIAL

For this example, the power  $P_d$  dissipated by unit area of material can be evaluated from the calculation of  $R_a$  and  $T_b$  by

$$P_d = (1 - |R_a|^2 - |T_b|^2)(\cos \theta / Z_c). \quad (67)$$

This power is related to an amplitude  $D$  of the displacement in the  $x_3$  direction, at the contact surface of air and layer (1), equal to

$$D = |(1 + R_a)/Z_a \omega|. \quad (68)$$

The reflection coefficient  $R_a$  is related to  $Z_a$  by

$$R_a = (Z_a - Z_c / \cos \theta) / (Z_a + Z_c / \cos \theta). \quad (69)$$

#### 5. SECOND EXAMPLE

The second stratified material is represented in Figure 5. It is composed of a porous layer bonded onto a layer of aluminium. The main parameters which describe the porous layer are given in Table 2.

The calculation of the matrix  $[T^p]$  in the context of Biot theory and a systematic use of the characteristic dimensions [10] is given in reference [1].

The thickness of the layer of aluminium is equal to 1 mm. The matrix  $[D]$  and the vector  $V$  of equation (56) for this material are

$$[D] = \begin{bmatrix} [I_{fs}^i] & [J_{fs}^i][T^s] & 0 & 0 \\ 0 & [I_{sp}] & [J_{sp}][T^p] & 0 \\ 0 & 0 & [I_{pf}^o] & [J_{pf}^o] \end{bmatrix}, \quad (70)$$

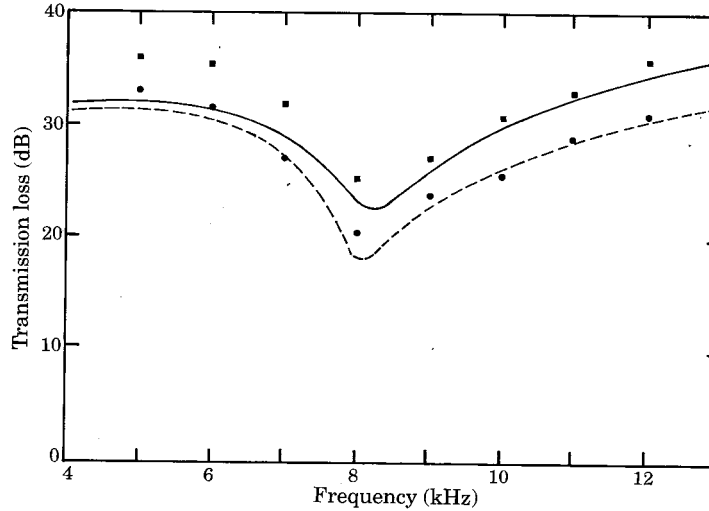


Figure 4. The transmission loss for the stratified material described in Table 1, and for the first layer of Table 1 without the viscoelastic layer and the aluminium foil. Stratified material: prediction —, measurement ■ ■ ■; layer 1: prediction ---, measurement ● ● ●.

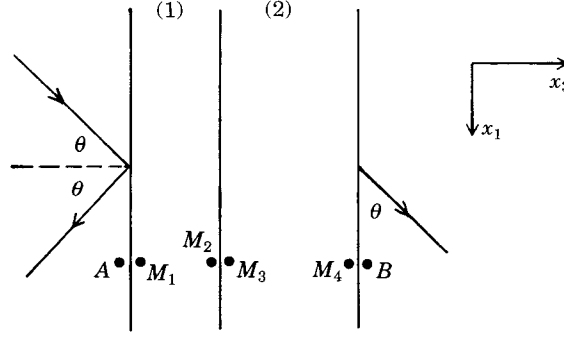


Figure 5. A porous foam (2) bonded on to a layer of aluminium (1)

$$V = [p(A), v^f(A), V_s(M_2), V_p(M_4), p(B), v^f(B)]. \quad (71)$$

The transmission coefficient  $T$  and the impedance  $Z_a$  can be calculated by equations (61) and (63). An approximate modelling of the aluminium layer previously used in similar problems [11–13] consists in considering, instead of the elastic solid layer, a plate subjected to flexural vibrations. Applying this approximation, at  $A$ ,  $M_1$ ,  $M_2$  and  $M_3$  yields the following equations. For the velocity components,

$$v_3^f(A) = v_3^s(M_1) = v_3^s(M_2) = v_3^s(M_3) = v_3^f(M_3), \quad (72)$$

$$v_1^s(M_1) = -v_1^s(M_2) = -v_1^s(M_3) = (e/2) dv_3^s(M_1)/dx_1, \quad (73)$$

where  $e$  is the thickness of the plate and  $dv_3(M_1)/dx_1$  is given by

$$dv_3^s(M_1)/dx_1 = -jk_1 v_3(M_3) \quad (74)$$

(the chosen time dependence is  $\exp(j\omega t)$ ).

The equation of motion of the plate is

$$\frac{D}{j\omega} \frac{d^4 v_3^s(M_2)}{dx_1^4} + \rho e j\omega v_3^s(M_2) = p(A) + \sigma_{33}^f(M_3) + \sigma_{33}^s(M_3) + \frac{e}{2} \frac{d\sigma_{13}^s(M_3)}{dx_1}, \quad (75)$$

where  $\rho$  is the density of the plate and  $D$  is the flexural rigidity given by

$$D = Ee^3/12(1 - \nu^2), \quad (76)$$

in which  $E$  and  $\nu$  are respectively Young's modulus and the Poisson ratio. The following equations relate the velocity components and the stress components at  $A$  and  $M_3$ :

$$v_3^f(M_3) = v_3^f(A), \quad v_3^s(M_3) = v_3^f(A), \quad v_1^s(M_3) = jk_1(e/2)v_3^f(A), \quad (77-79)$$

TABLE 2

*Parameters for the porous material*

Thickness (m)	$5 \times 10^{-3}$
Density (kg/m <sup>3</sup> )	100
Tortuosity	1
Flow resistivity (N m <sup>-4</sup> s)	$10^4$
Frame shear modulus (Pa)	$5 \times 10^7 + j10^6$
Poisson ratio	0
Porosity	0.99
Characteristic dimensions (m)	$\Lambda = 1.2 \times 10^{-4}$ , $\Lambda' = 1.2 \times 10^{-4}$

$$\sigma_{33}^f(M_3) + \sigma_{33}^s(M_3) - j(k_1 e/2)\sigma_{13}^s(M_3) = \{\rho e j\omega + (D/j\omega)k_1^4\}v_3(A) - p(A). \quad (80)$$

From equations (77)–(80), one obtains the following interface matrices  $[I_{f,pl}]$  and  $[J_{f,pl}]$  which relate  $V_f$  at  $A$  and  $V_p$  at  $M_3$ :

$$[I_{f,pl}] = \begin{bmatrix} 0 & jk_1 e/2 \\ 0 & 1 \\ 0 & 1 \\ -1 & \rho e j\omega + D(k_1)^4/j\omega \end{bmatrix}, \quad [J_{f,pl}] = - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -j \frac{k_1 e}{2} & 1 \end{bmatrix}. \quad (81, 82)$$

The matrix  $[D]$  and the vector  $V$  with this modelling are

$$[D] = \begin{bmatrix} [I_{f,pl}] & [J_{f,pl}] & [T^p] & 0 \\ 0 & [I_{p,f}] & & [J_{p,f}] \end{bmatrix}, \quad (83)$$

$$V = [p(A), v^f(A), V_s(M_4), p(B), v^f(B)]^T. \quad (84)$$

This second model fails when the porous layer is very stiff, as the mid-plane of the layer of aluminium is no longer a neutral layer. The robustness of the model can be evaluated by arbitrarily increasing the stiffness of the foam and comparing the predicted transmission loss obtained by both models.

The transmissions predicted by both models are represented in Figure 6 for a very large stiffness of the foam, with a shear modulus  $G$  and a Poisson coefficient given by  $G = 10^8(1 + j0.1)$  Pa and  $\nu = 0.4$ .

The small discrepancy appears for small values of  $G$ , and is not created by the stiffness of the foam but by using a plate model for an elastic solid layer. It appears that the second model can be used with confidence for most stratified materials made up of a porous layer bonded on to a metal plate.

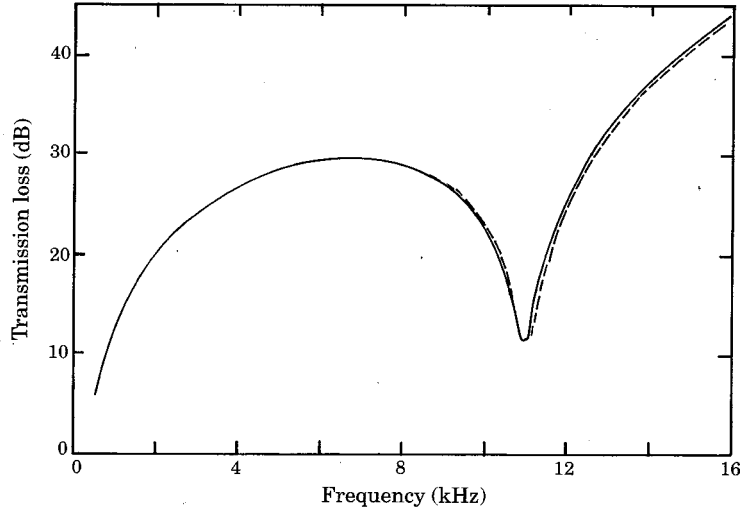


Figure 6. The computed transmission loss for the stratified material made up of a porous layer bonded on to an aluminium layer. Aluminium layer modelled as an elastic layer, ---; aluminium layer modelled as a plate, —.

## 6. CONCLUSIONS

A simple method of modelling sound propagation in stratified materials has been developed. A program derived from this method for the prediction of sound transmission and reflection of plane waves by these materials has been created, and used for two different layered media. The large range of materials which can be studied with the same program demonstrates the versatility of the method.

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## APPENDIX

A plane wave impinges upon the material at an angle of incidence  $\theta_a$  on the left-hand side in Figure A1 in the presence of a uniform steady flow  $V_a$  in the semi-infinite layer of air, parallel to the surface. Losses in air are disregarded and the perfect fluid, linearized equations, are

$$\rho_0(\partial \mathbf{v} / \partial t + (\mathbf{V}_a \cdot \nabla) \mathbf{v}) = -\nabla p, \quad (1/K_a)(\partial p / \partial t + (\mathbf{V}_a \cdot \nabla) p) = -\nabla \cdot \mathbf{v}, \quad (\text{A1})$$

where  $\rho_0$  is the ambient density,  $K_a$  is the adiabatic bulk modulus of air, and  $\mathbf{v}$  and  $p$  are the acoustic excess velocity and pressure. Consider a plane wave component

$$\mathbf{e}^{j(\omega t - \mathbf{k}(\theta) \cdot \mathbf{r})}, \quad (\text{A2})$$

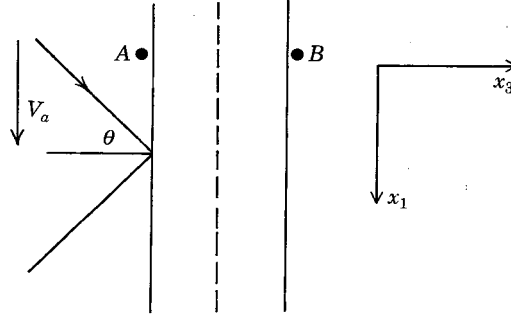


Figure A1. A stratified material in air with a uniform steady flow of velocity  $V_a$  at its left-hand side.

at angle  $\theta$  with  $x_3$ ,

$$\mathbf{k}(\theta) = \tilde{k} \sin \theta \hat{\mathbf{x}}_1 + \tilde{k} \cos \theta \hat{\mathbf{x}}_2, \quad \mathbf{r} = x_1 \hat{\mathbf{x}}_1 + x_2 \hat{\mathbf{x}}_2, \quad (\text{A3})$$

where  $\hat{\mathbf{x}}_1$  and  $\hat{\mathbf{x}}_2$  are the unit vectors in the directions  $x_1$  and  $x_2$  respectively. For such a component at incidence angle  $\theta$ , the two equations (A1) are equivalent to

$$\tilde{\rho}_0 \partial \mathbf{v} / \partial t = -\nabla p, \quad (1/\tilde{K}_a) \partial p / \partial t = -\nabla \cdot \mathbf{v}, \quad (\text{A4})$$

with

$$\tilde{\rho}_0 = \rho_0 \frac{1}{1 + (V_a/c) \sin \theta}, \quad \frac{1}{\tilde{K}_a} = \frac{1}{K_a} \frac{1}{1 + (V_a/c) \sin \theta}, \quad (\text{A5})$$

where  $c = (K_a/\rho_0)^{1/2}$  is the adiabatic speed of sound. This can be obtained by introducing a velocity potential  $\Phi$ , which behaves as indicated in equation (A2), and is such that

$$\mathbf{v} = \nabla \Phi. \quad (\text{A6})$$

Then, by using equation (A1) one obtains

$$p = -\rho_0 (\partial \Phi / \partial t + (\mathbf{V}_a \cdot \nabla) \Phi), \quad (\text{A7})$$

and a further use of equations (A2) and (A1) yields equation (A5). Hence, the characteristic impedance

$$\tilde{Z} = (\tilde{\rho}_0 \tilde{K}_a)^{1/2} = (\rho_0 K_a)^{1/2} = Z_c \quad (\text{A8})$$

is independent of  $V_a$  and  $\theta$ . The effect of the mean flow is only to provide a propagation constant and speed of sound which depend on  $V_a$  and  $\theta$ :

$$\tilde{k} = \omega (\tilde{\rho}_0 / \tilde{K}_a)^{1/2} = (\omega/c) / \{1 + (V_a/c) \sin \theta\}, \quad \tilde{c} = c \{1 + (V_a/c) \sin \theta\}. \quad (\text{A9})$$

In particular, the projection  $k_1$  of the wavenumber is

$$k_1 = (\omega/\tilde{c}) \sin \theta = (\omega/c) \sin \theta / \{1 + (V_a/c) \sin \theta\}. \quad (\text{A10})$$

One can now examine the validity of equations (1)–(6). The (invertible) transfer matrix in equation (1) describes the linear response of the black box when some pressure  $p$  and normal flow  $v_3^f$  that are characterized by the temporal and spatial periodicities  $2\pi/\omega$  and  $2\pi/k_1$ , are imposed at the surface. If boundary layer effects that may be induced by the external mean flow are neglected, no change is brought to this generic problem, and  $p$  and  $v_3^f$  in equation (1) are still acoustic quantities. However, for determining the new value (at angle  $\theta$ , parallel flow  $V_a$  and angular frequency  $\omega$ ) of the spatial periodicity  $2\pi/k_1$ , equation

(A9) must be used and substituted into equation (7) to obtain the suitable transfer matrix coefficients. As noted in section 1, equations (2) and (3) apply with the proper angles  $\theta_a$  and  $\theta_b$ . It follows that

$$T_a = (ad - bc) \frac{2Z_c^{-1} \cos \theta_b}{c + dZ_c^{-1} \cos \theta_b + Z_c^{-1} \cos \theta_b (a + bZ_c^{-1} \cos \theta_b)}, \quad (\text{A11})$$

$$T_b = \frac{2Z_c^{-1} \cos \theta_a}{c + dZ_d^{-1} \cos \theta_a + Z_c^{-1} \cos \theta_a (a + bZ_c^{-1} \cos \theta_a)}. \quad (\text{A12})$$

Obviously,  $ad - bc = 1$  since  $a, b, c$  and  $d$  correspond to the usual transfer matrix in the absence of flow, for an angle  $\theta^*$  such that

$$\sin \theta^* = \sin \theta_a / \{1 + (V_a/c) \sin \theta_a\}. \quad (\text{A13})$$

The transmissions in the normal directions coincide:

$$T_a \cos \theta_a = T_b \cos \theta_b. \quad (\text{A14})$$