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Explicit approximation of the wavenumber for lined ducts

Maaz Farooqui,¹ Yves Aurégan,¹ and Vincent Pagneux¹

Laboratoire d'Acoustique de l'Université du Mans, Centre National de la Recherche Scientifique (CNRS), Le Mans Université, Avenue Olivier Messiaen, 72085 Le Mans Cedex 9, France

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Abstract: For acoustic waves in lined ducts, at given frequencies, the dispersion relation leads to a transcendental equation for the wavenumber that has to be solved by numerical methods. Based on Eckart explicit expression initially derived for water waves, accurate explicit approximations are proposed for the wavenumber of the fundamental mode in lined ducts. While Eckart expression is 5 % accurate, some improved approximations can reach maximum relative error of less than 10^{-8} . The cases with small dissipation part in the admittance of the liner and/or axisymmetric ducts are also considered.

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1. Introduction

In a duct with a locally reacting liner, a waveguide with admittance boundary conditions at the wall, the dispersion relation allows to calculate the wavenumbers as a function of the frequency and of the liner admittance¹. Since this dispersion relationship leads to a transcendental equation, there is no closed form expression for the wavenumber and iterative numerical methods are most often used. In view of the numerous applications of lined ducts²⁻¹¹, it could be very interesting to have an accurate explicit expression of the wavenumber rather than a numerical value.

In the field of water waves, Eckart¹²¹ gave an approximated value of the wavenumber with 5% accuracy on the whole frequency range. Thereafter, other explicit approximations, extremely accurate but also more complex, were proposed for water waves¹⁴⁻¹⁶. To the best of our knowledge, this type of explicit approximations of the wavenumber has not been used in acoustics although the dispersion relation of water waves and acoustic waves in lined ducts are very similar. In this letter, we first show how the Eckart formula and some of its improvements can be used to compute, with an accuracy of up to $2.8 \times 10^{-7}\%$, the wavenumber of the fundamental mode in a lined duct with a purely reactive admittance. Then, we present an extension of these explicit approximations from the two-dimensional (2D) case to the axisymmetric case. Finally, we show that we can also predict the wavenumbers when the real part of the admittance is slightly negative, modelling moderate dissipation in the liner. It should be noted that the proposed explicit approximations are not valid for multimodal propagation and/or highly dissipative liners.

2. Application to Non-Dissipative admittance

2.1 2D case

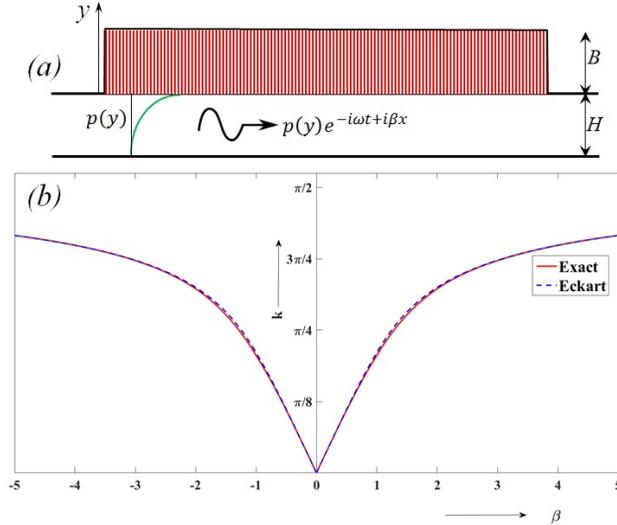


Fig. 1. (color online) (a) schematic description of the 2D problem for duct of height H with lining thickness B . (b) Reduced frequency k as a function of the wavenumber β for lined duct. The exact solution is plotted as a continuous line while the Eckart approximation is in dashed line ($B = H$, the liner resonance is at $k = \pi/2$).

Let us start by considering the sound propagation in a 2D waveguide where the lower wall is rigid while the upper wall is compliant and described by a admittance Y (Fig. 1(a)). When the distances are non-dimensional by the height of the channel H , the Helmholtz equation, governing the propagation of the acoustic pressure p , is $\Delta p + k^2 p = 0$, where $k = \omega H/c_0$ is the reduced frequency, ω is the frequency and c_0 is the sound velocity. The boundary conditions are $\partial_y p = 0$ at $y = 0$, for the rigid wall and $\partial_y p = Y p$ at $y = 1$, for the

wall with liner. For a uniform admittance Y , the modal solution can be written under the form $p = A \cosh(\alpha y) \exp(i(-\omega t + \beta x))$, where

$$\beta^2 = \alpha^2 + k^2, \quad (1)$$

leading to the dispersion relation:

$$Y = \alpha \tanh(\alpha). \quad (2)$$

To solve this equation with α as the unknown is the central subject of this paper. Of course, once α is found it yields the wavenumber β through Eq. (1). When $Y > 0$, a nice explicit approximation (coming from Eq. (2)) of α as a function of the admittance Y is given by the Eckart formula¹²

$$\alpha_E = \frac{Y}{\sqrt{\tanh(Y)}}. \quad (3)$$

This relationship is valid for the fundamental mode (α real) and it takes into account the two limit cases: $\alpha \ll 1$ for which $Y \simeq \alpha^2$ and $\alpha \rightarrow +\infty$ for which $Y \simeq \alpha$, giving a good approximation between these two limits. Eq. (3) holds for acoustic wave propagation in a 2D lined duct up to the resonance frequency of the liner (where $Y \rightarrow +\infty$). This is illustrated in Fig. 1(b) where α and then β are computed exactly **as well as** by the Eckart approximation. In this case, the admittance is given by $Y = k \tan(kB)$ where B is the height of the liner made of lossless tubes non-dimensioned by H .

In the following, the error ε on an approximate transverse wavenumber α_{app} is defined using $\varepsilon = |(\alpha_{app} - \hat{\alpha})/\hat{\alpha}|$ where $\hat{\alpha}$ is the exact value². The error of the Eckart approximation

is displayed in blue dashed line in Fig. 2(a) as a function of the admittance Y . It can be seen that the error ε does not exceed 5%.

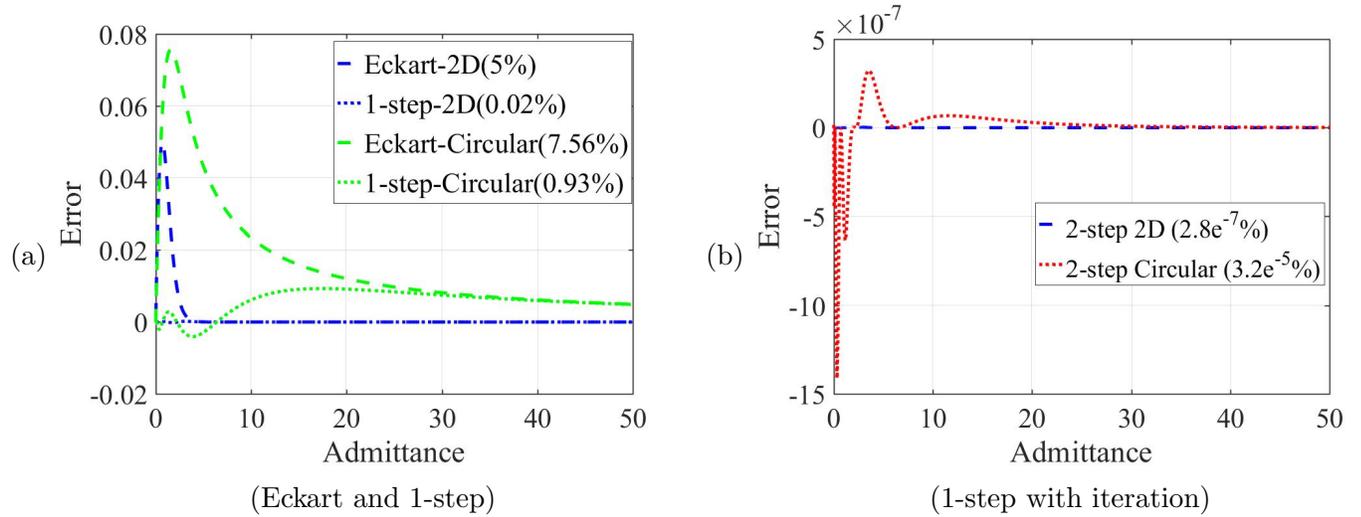


Fig. 2. (color online) Error ε as a function of the real admittance Y . (a) Eckart formula and 1-Step approximation (b) First Newton iteration of 1-Step approximation

Another and much better approximation can be obtained by an empirical fit of the error of the Eckart approximation¹⁶ which leads to explicit expression

$$\alpha_0 = \frac{Y + Y^{1.986} e^{-(1.863 + 1.198Y^{1.366})}}{\sqrt{\tanh(Y)}} \quad (4)$$

that is called in the following the 1-Step approximation. Using this relation, the error is reduced to $\varepsilon = 0.02\%$ and is shown in blue dotted line on Fig. 2(a) that is indistinguishable from zero. The four coefficients of Eq. (4) were used as initial guess band (± 1) in order to obtain formulas of the cases discussed later.

To increase further the accuracy of the prediction, it also possible to use the first iteration of the Newton method that can be written explicitly with a particularly simple

expression. Considering α_0 as an initial value, we obtain the 2-Step approximation¹⁵ :

$$\alpha_* = \frac{\alpha_0^2 + Y \cosh^2 \alpha_0}{\alpha_0 + \sinh \alpha_0 \cosh \alpha_0}. \quad (5)$$

The error in α_* is around $2.8 \times 10^{-7}\%$ and is displayed in Fig 2(b) in dashed line. If one intends to apply directly, the Newton's method (Eq. (5)) with Eckart approximation (α_E) instead of α_0 , the error will be around $8.4 \times 10^{-3}\%$

2.2 Axisymmetric circular case

Often, practical situations involve circular ducts and the same type of work as in the 2D case can be done for axisymmetric circular ducts with liner. In dimensionless form, the equation governing acoustic pressure p in axisymmetric circular ducts in the transverse direction is $p'' + \frac{1}{r}p' + (k^2 - \beta^2)p = 0$, where $k = \omega R/c_0$ is the reduced frequency and R is the duct radius. The admittance boundary condition is $\frac{\partial p}{\partial r} = Yp$ for $r = 1$. For a uniform admittance, the solution is then searched under the form $p = AI_0(\alpha r) \exp(i(-\omega t + \beta x))$ where $\alpha^2 = \beta^2 - k^2$, leading to the dispersion relation:

$$Y = \alpha \frac{I_1(\alpha)}{I_0(\alpha)} \quad (6)$$

where $I_{0,1}$ are the modified Bessel function of order 0 and 1. Following the idea of Eckart we obtain the new approximation

$$\alpha_E = \frac{Y}{\sqrt{I_1(Y)/I_0(Y)}}. \quad (7)$$

The maximal error for $Y > 0$ is then 7.56%. (see Fig. 2(a)).

As in 2D case, one can achieve a better accuracy by a 1-Step approximation α_0

$$\alpha_0 = \frac{Y + Y^{2.5285} e^{-(.0216+2.6034Y^{0.4615})}}{\sqrt{I_1(Y)/I_0(Y)}} \quad (8)$$

with an error that is around 0.93%. The corresponding 2-Step approximation α_* has a $3.2 \times 10^{-5}\%$ error as in Fig. 2(b):

$$\alpha_* = \alpha_0 + \frac{J_0(\alpha_0)(Y I_0(\alpha_0) - \alpha_0 I_1(\alpha_0))}{\alpha_0(I_0(\alpha_0)^2 - I_1(\alpha_0)^2)} \quad (9)$$

If one intends to apply directly the Newton's method (Eq. (9)) with Eckart approximation (α_E) instead of α_0 , the error will be around 0.01%.

3. Application in dissipative Cases

3.1 2D case

The Eckart approximation, which is valid for Y real and positive, can be extended to cases of great interest for acoustics: the cases when the real part of the admittance is slightly negative corresponding to a moderate dissipation (given by a positive imaginary part of Y). The results of this continuation of the Eckart approximation in the complex plane is displayed in Fig. 3 for the 2D case. It can be seen that this approximation is accurate around $Y = 0$ and for $\Re(Y)$ large. When $\Re(Y)$ is negative the error increases quite rapidly with $|\Re(Y)|$. The effect of the dissipation is weak when $\Re(Y)$ is positive and large but adding dissipation significantly increases the error when $\Re(Y)$ is low or negative. Then, the worst error is 20% corresponds to the largest negative $\Re(Y)$ and the largest $\Im(Y)$ that we have considered (Fig. 4(a)). Following a procedure similar to the non-dissipative case, an improved 1-Step

approximation is found as

$$\hat{\alpha}_0 = \frac{Y + Y^{2.0987} e^{-(1.8340 + 0.6507Y^{1.1891})}}{\sqrt{\tanh(Y)}} \quad (10)$$

with a maximal error of 7% (Fig. 4(c)). The corresponding 2-Step approximation α_* (Eq. (5)) has only 0.28% error (Fig. 4(e)) for $\Re(Y) \in [-0.5, \infty]$, $\Im(Y) \in [0, 0.5]$. Applying (Eq. (5)) with Eckart approximation α_E (Eq. (3)) instead of $\hat{\alpha}_0$, the error is around 3.1%.

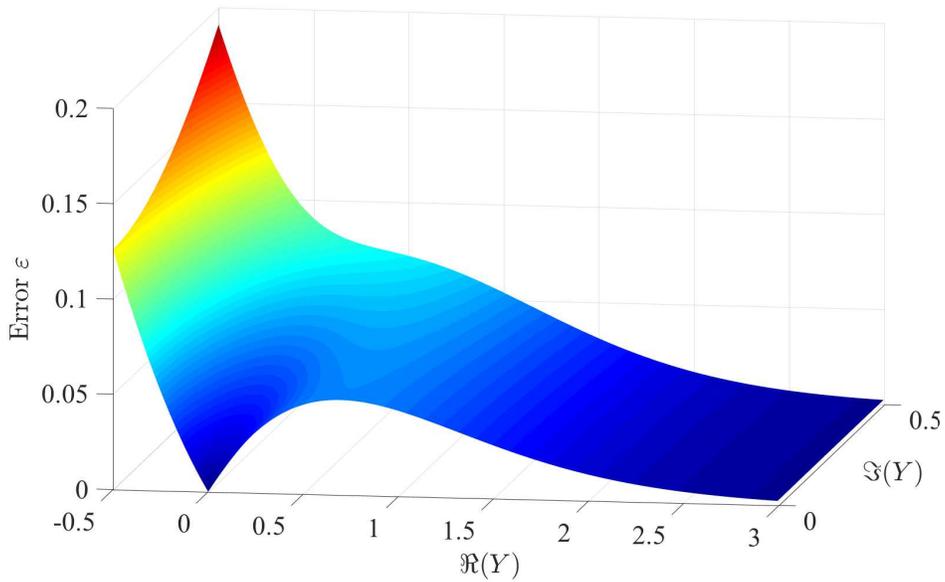


Fig. 3. (color online) Error of the Eckart approximation (Eq. (3)) in the complex plane of the admittance Y with small dissipation.

3.2 2D Circular case

Following the same lines as previously, the Eckart approximation (Eq. (7)) error is around 11.5% as shown in Fig. 4(b). Then, Fig. 4(d) shows that, for $Y_{Real} \in [-0.5, \infty]$, $Y_{Imag} \in [0, .5]$,

one can achieve 6.33% error using the approximation

$$\hat{\alpha}_0 = \frac{Y + Y^{2.0583} e^{-(2.0334 + 0.3717Y^{1.0407})}}{\sqrt{I_1(Y)/I_0(Y)}}. \quad (11)$$

The error in $\hat{\alpha}_*$ (Eq. (9)) will be 0.02% as in Fig. 4(f). On directly applying Newton method (Eq. (9)) with Eckart approximation α_E (Eq. (7)) instead of $\hat{\alpha}_0$, the error will be around 1%.

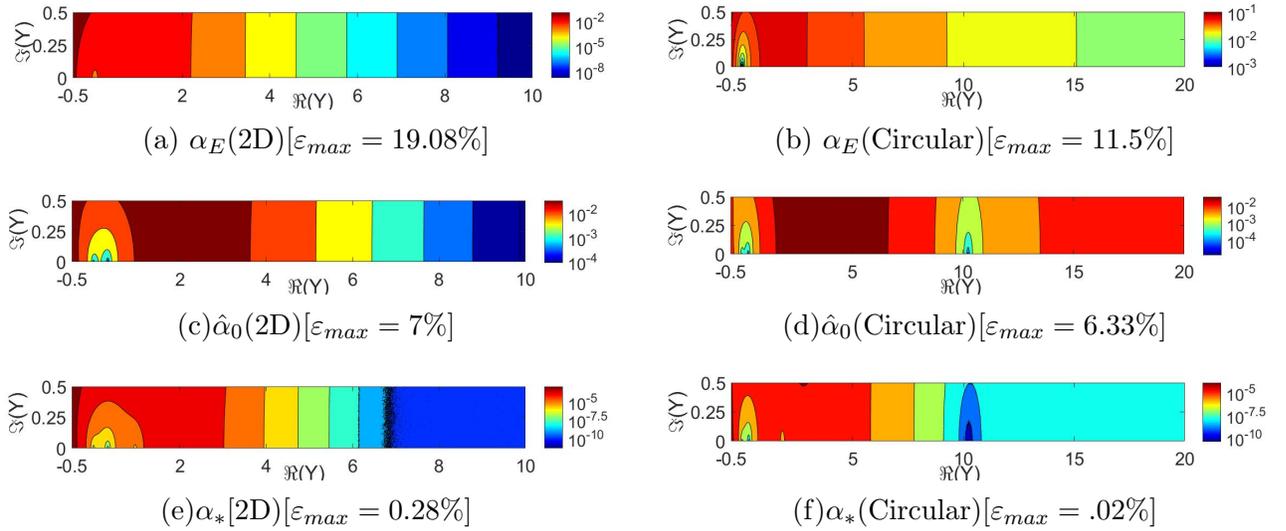


Fig. 4. (color online) Errors with respect to exact solution of Complex Admittance Y using

$Y = \alpha \tanh \alpha$ for 2D ducts and $Y = \alpha \frac{I_1(\alpha)}{I_0(\alpha)}$ for Circular ducts (a,b) α_E from Eckart formula (c,d)

$\hat{\alpha}_0$ from 1-Step proposition (e,f) α_* from first Newton iteration of 1-Step proposition

4. Conclusion

Explicit approximations for wavenumbers have been proposed to analyze the propagation in a lined duct. The very simple Eckart expression approximates with a reasonable accuracy the exact dispersion relation when the dissipation is weak and the imaginary part of the

admittance is not too negative. It has been shown that the accuracy can be improved by introducing empirical corrections and/or by using the explicit first iteration of the Newton method. These explicit approximations may be used easily for practical purposes: explicit one-mode determination can simplify models based on low frequency acoustic wave propagation and several applications, in 2D ducts²⁻⁷ as well as Axisymmetric ducts⁷⁻¹¹, can be made simpler using this type of approximation at least as a starting point.

Acknowledgments

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References and links

- ¹P. M. Morse and K. U. Ingard, *Theoretical acoustics* (Princeton university press, 1968), chap: Sound waves in Ducts and Rooms, 467–599.
- ²Y. Aurégan and V. Pagneux, “Slow sound in lined flow ducts,” *The Journal of the Acoustical Society of America* **138**(2), 605–613 (2015).
- ³W. Eversman, “The effect of mach number on the tuning of an acoustic lining in a flow duct,” *The Journal of the Acoustical Society of America* **48**(2A), 425–428 (1970).
- ⁴M. Jones, T. Parrott, and W. Watson, “Comparison of acoustic impedance reduction techniques for locally-reacting liners,” in *9th AIAA/CEAS Aeroacoustics Conference and Exhibit* (2003), p. 3306.

- ⁵M. Farooqui, T. Elnady, and M. Åbom, “Measurement of perforate impedance with grazing flow on both sides,” in *22nd AIAA/CEAS Aeroacoustics Conference* (2016), p. 2853.
- ⁶M. Farooqui, Y. Aurégan, and V. Pagneux, “Guiding acoustic waves over obstacles using linear surface modes,” in *11th European Congress and Exposition on Noise Control Engineering* (2018), pp. 45–48.
- ⁷A. H. Nayfeh and D. P. Telionis, “Acoustic propagation in ducts with varying cross sections,” *The Journal of the Acoustical Society of America* **54**(6), 1654–1661 (1973).
- ⁸L. M. Campos and J. M. Oliveira, “On the acoustic modes in a cylindrical duct with an arbitrary wall impedance distribution,” *The Journal of the Acoustical Society of America* **116**(6), 3336–3347 (2004).
- ⁹W. Bi, V. Pagneux, D. Lafarge, and Y. Aurégan, “An improved multimodal method for sound propagation in nonuniform lined ducts,” *The Journal of the Acoustical Society of America* **122**(1), 280–290 (2007).
- ¹⁰S. W. Rienstra, “A classification of duct modes based on surface waves,” *Wave motion* **37**(2), 119–135 (2003).
- ¹¹P. Vaidya, “The propagation of sound in ducts lined with circumferentially non-uniform admittance of the form $\eta_0 + \eta_q \exp(iq\theta)$,” *Journal of Sound and Vibration* **100**(4), 463–475 (1985).
- ¹²C. Eckart, “The propagation of gravity waves from deep to shallow water,” in *Gravity waves* (1952), p. 165.

¹³For any expression of the form, $Y = \alpha f(\alpha)$ such as $f(\alpha) = \alpha$ for α close to 0 and $f(\alpha) = 1$ for large α , Eckart¹² proposed an explicit approximation of α given by $\alpha = Y/\sqrt{f(Y)}$. For water waves, this explicit definition approximates α with 5% accuracy.

¹⁴S. Beji, “Improved explicit approximation of linear dispersion relationship for gravity waves,” *Coastal Engineering* **73**, 11–12 (2013).

¹⁵G. Simarro and A. Orfila, “Improved explicit approximation of linear dispersion relationship for gravity waves: Another discussion,” *Coastal Engineering* **80**, 15 (2013).

¹⁶A. R. Vatankhah and Z. Aghashariatmadari, “Improved explicit approximation of linear dispersion relationship for gravity waves: Comment on another discussion,” *Coastal engineering* **81**, 30–31 (2013).

¹⁷To evaluate this error, we start from a given $\hat{\alpha}$. From the dispersion relation Eq. 2, we obtain the associated admittance Y . Introducing this admittance in the approximated relation Eq. 3, we obtain the approximated α_E and we can compute the error.

References and links

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